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Arc-Disjoint Path Pair (APP) Problem

Nang Kham Maing

Abstract

Let $G(V,A)$ be a directed network containing $n$ nodes $v \in V$ and $m$ arcs $(i,j) \in A$, each with a non-negative length. In this paper arc-disjoint path pair problem is considered, which is to find a pair of arc-disjoint paths from a source to a destination in a directed network. We described Bhandari's algorithm to solve arc-disjoint path pair problem and also proved the correctness of algorithm.

Key words: Directed network, Shortest paths, Arc-disjoint path.

Introduction

Disjoint paths are used in communication networks for reliability of transmission between a given source and destination. Paths between a given pair of source and destination nodes in a network are called arc-disjoint if they have no common arcs. In this paper, we consider the problem of finding a pair of arc-disjoint paths between a pair of nodes in a directed network.

Firstly, we introduced arc-disjoint path pair problem and some basic definitions and notations are presented. Secondly, Bhandari's algorithm for solving arc-disjoint path pair problem is described. Finally, the correctness of Bhandari's algorithm is proved and we also prove that the solution produced with Bhandari's algorithm is optimal.

Statement of Arc-Disjoint Path Pair Problem

Given a network $G(V,A)$ for a source-destination pair $(s,t)$, find a set of two paths $P_1$ and $P_2$, such that $P_1 \cap P_2 = \emptyset$ and the total length $l(P_1) + l(P_2)$ is minimized.

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Some Basic Definitions and Notations on Arc-Disjoint Path Pair Problem

An arc from node \( i \) to node \( j \) is represented as \((i, j)\), \( i, j \in V \). Each arc \((i, j)\) has a numerical length (weight) \( w_{ij} \). We assume that only non-negative arc lengths are assigned to each arc. However, in the process of computing disjoint paths, negative arc lengths may be assigned to arcs.

Let \( G (V, A) \) be a network, a path \( P \), between a source \( s \) and destination \( t \) is considered to be a set of arcs that compose this path. With a slight abuse of notation, we choose \( P \) to denote the path as well as its arc set. The length of a path \( l(P) \) is computed as \( l(P) = \sum_{(i,j) \in P} w_{ij} \).

We define the total length of two paths as \( l(P_1) + l(P_2) \).

Let \( P \) be a path from \( s \) to \( t \), \( q \) is said to be a sub-path of \( P \) if it coincides with \( P \) from \( s \) until \( y \).

The shortest path between two nodes \( i \) and \( j \) is the path joining \( i \) to \( j \), that has minimum length.

The intersection of paths \( P_1 \) and \( P_2 \) is the set of arcs which contains both \( P_1 \) and \( P_2 \),

\[
P_1 \cap P_2 = \{(i, j) \in A \mid (i, j) \in P_1 \text{ and } (i, j) \in P_2\}.
\]

The union of paths \( P_1 \) and \( P_2 \) is the set of arcs which contains either \( P_1 \) or \( P_2 \), i.e., \( P_1 \cup P_2 = \{(i, j) \in A \mid (i, j) \in P_1 \text{ or } (i, j) \in P_2\} \).

If path \( P_1 \) is arc-disjoint with \( P_2 \), there is no common arc element in the arc set representing each path and \( P_1 \cap P_2 = \emptyset \), else \( P_1 \cap P_2 \neq \emptyset \).

If path \( P_1 \) is arc-disjoint with \( P_2 \), i.e., \( P_1 \cap P_2 = \emptyset \), we have \( l(P_1 \cup P_2) = l(P_1) + l(P_2) \).

The path which is obtained by reversing the direction and the sign of the arc lengths of each arc on the path \( P_1 \) between \( s \) and \( t \) is called the path \(-P_1\) directed from \( t \) to \( s \).
Fig. 1. Paths $P_1$ and $-P_1$

$P_1 \sim P_2$ is a set, which consists of the $P_1$ arcs whose reversed arcs appear on $P_2$ and vice versa,
i.e., $P_1 \sim P_2 = \{(i, j) \text{ and } (j, i) \mid (i, j) \in P_1 \text{ and } (j, i) \in P_2\}$.

A modified network $G(V, A')$ is a network which is created by reversing the direction and the sign of the length of each arc on the shortest path $P_1$ from node $s$ to node $t$ in a directed network $G(V, A)$.

In all the figures, bold lines represent arcs on the shortest path(s) in a network or its corresponding modified network, dashed lines represent reversed arcs which do not exist in the original network and bold dashed lines represent such reversed arcs that appear on the shortest path.

**Bhandari's Algorithm**

The steps of the Bhandari's algorithm for solving arc-disjoint path pair problem are as follows:

Given a directed network $G(V, A)$, for a source-destination pair $(s, t)$,

**Step 1.** Find the shortest path $P_1$ from node $s$ to node $t$;

**Step 2.** Replace $P_1$ with $-P_1$, a modified network $G(V, A')$ is created;

**Step 3.** Find a shortest path $P_2$ from node $s$ to node $t$ in the modified network $G(V, A')$, if $P_2$ does not exist, then stop;

**Step 4.** Take the union of $P_1$ and $P_2$, remove from the union the arc set which consists of the $P_1$ arcs whose reversed arcs appear in $P_2$ and
vice versa, then group the remaining arcs into two paths \( P_1' \) and \( P_2' \), i.e., \( P_1' \cup P_2' = (P_1 \cup P_2) \setminus (P_1 \cap P_2) \)

We will explain the steps of Bhandari’s algorithm with examples in Fig. 2. and Fig. 3.

**Example (1)**

We are required to find a set of two shortest disjoint paths between \( s \) and \( t \).

In Step 1, the shortest path from \( s \) to \( t \) is found as \( P_1 = sact \), with minimum length 6. In Step 2, a modified network \( G (V, A') \) is created by replacing \( P_1 \) with \(-P_1\). In Step 3, the shortest path in the modified network \( P_2 = scabt \) has length 10. In Step 4, \( P_1 \cap P_2 = \{(a, c), (c, a)\} \) is removed from the union \( P_1 \cup P_2 \). The solution set of disjoint paths \( \{P_1', P_2'\} = \{sabt, stc\} \) is obtained. The total length of this path set equals \( 8 + 8 = 16 \), which is exactly the minimal total length of two arc-disjoint paths in this network.
Example (2)

Bhandari's algorithm is exemplified with the network in Fig. 3.

In Step 1, the shortest path from s to t is found as \( P_1 = scdt \), with minimum length 4. In Step 2, a modified network \( G(V, A') \) is created by replacing \( P_1 \) with \( -P_1 \). In Step 3, the shortest path in the modified network \( P_2 = sabt \) has length 7. In Step 4, \( P_1 \sim P_2 = \emptyset \). The solution set of disjoint paths \( \{P_1', P_2'\} = \{scdt, sabt\} \) is obtained. The total length of this path set equals \( 4 + 7 = 11 \) which is exactly the minimal total length of two arc-disjoint paths in this network.

Correctness of Bhandari's Algorithm

We will first show that the optimal solution set of Bhandari's algorithm is based on the shortest path.

Given a network \( G(V, A) \) and a pair of source-destination nodes \( (s, t) \), the relation between a set of two arc-disjoint paths \( \{P_1', P_2'\} \) and the shortest path \( P_1 \) belongs to one of the following types.
Lemma (1)

Given a directed network $G (V, A)$ and a source-destination pair $(s, t)$, if the optimal set $\{P'_1, P'_2\}$ of APP exists, $P'_1 \cup P'_2$ must contain either the first shortest path $P_1$ itself or some $P_1$ arcs on each of its two paths.

Proof.

If $P'_1 \cup P'_2$ is of type (4), then each path in $\{P'_1, P'_2\}$ is arc-disjoint with $P_1$. As $P_1$ is the shortest path, both $\{P_1, P'_1\}$ and $\{P_1, P'_2\}$ have a total length shorter than $\{P'_1, P'_2\}$. Hence the optimal set $\{P'_1, P'_2\}$ cannot be of type (4) and $P'_1 \cup P'_2$ must contain some or all $P_1$ arcs to be the optimal set.

If $P'_1 \cup P'_2$ is of type (3), only one path in $P'_1 \cup P'_2$ contains some $P_1$ arcs, without loss of generality, suppose $P'_1$ contains some $P_1$ arcs and the other path $P'_2$ is arc-disjoint with $P_1$, then $\{P_1, P'_2\}$ is a set which is shorter than $\{P'_1, P'_2\}$. Hence the optimal set $\{P'_1, P'_2\}$ cannot be of type (3).

Therefore, if the optimal set $\{P'_1, P'_2\}$ exists, $P'_1 \cup P'_2$ must be either of type (1) or (2). $\blacksquare$

Definition (1)

The logical difference set $P_2 - P_1$ also can be computed as

$$P_2 - P_1 = \{(i, j)| (i, j) \in P_2 \setminus (P_2 \cap P_1)\} \cup \{(j, i)| (i, j) \in P_1 \setminus (P_2 \cap P_1)\}$$

which means that if an arc $(i, j)$ of $P_2$ does not appear on $P_1$, then this arc belongs to the difference set $P_2 - P_1$ and if an arc $(i, j)$ of $P_1$ does not
appear on $P_2$, then its direction reversed arc $(j, i)$ belongs to the difference set $P_2 - P_1$ with an arc length $w_{ij} = -w_{ji}$.

**Remark**

In set theory, the difference operation is defined as $P_2 - P_1 = P_2 \setminus (P_1 \cap P_2)$ and the symmetric difference operation is defined as $P_2 - P_1 = (P_2 \cup P_1) \setminus (P_1 \cap P_2)$. The concept of logical difference set in this paper resembles the symmetric difference set but it is not the same.

**Lemma (2)**

Let us denote $O_I = (P'_1 \cup P'_2) \sim (-P_I)$, which means that the set $O_I$ consists of each $P_1$ arc in the union of $P_2 \cup P_1$ and its corresponding $-P_1$ arc. Then $O_I = P'_1 \cup P'_2 \cap P_1$, and $l(O_I) = 0$.

**Proof.**

$$O_I = (P'_1 \cup P'_2) \sim (-P_I)$$

$$= \{(i, j), (i, j)\mid (i, j) \in P'_1 \cup P'_2 \text{ and } (j, i) \in -P_I\}$$

$$= \{(i, j), (i, j)\mid (i, j) \in P'_1 \cup P'_2 \text{ and } (i, j) \in P_1\}$$

$$= \{(i, j)\mid (i, j) \in P'_1 \cup P'_2 \cap P_1\}$$

$$= P'_1 \cup P'_2 \cap P_1.$$

$$l(O_I) = 0$$

because the set $O_I$ consists of cycles with zero length, each consisting of a pair of opposite arcs $P_1$ and $-P_1$.

**Lemma (3)**

The logical difference set between $P'_1 \cup P'_2$ and $P_1$ is

$$(P'_1 \cup P'_2) - P_1 = P'_1 \cup P'_2 \cup (-P_I) \setminus O_I.$$

**Proof.**

$$(P'_1 \cup P'_2) - P_1 = \{(i, j)\mid (i, j) \in P'_1 \cup P'_2 \setminus (P'_1 \cup P'_2) \cap P_1\} \cup$$

$$\{(j, i)\mid (j, i) \in P_1 \setminus (P'_1 \cup P'_2) \cap P_1\}$$

$$= \{(i, j)\mid (i, j) \in P'_1 \cup P'_2 \cup (-P_I) \backslash (P'_1 \cup P'_2) \cap P_1\}$$

$$= \{(i, j)\mid (i, j) \in P'_1 \cup P'_2 \cup (-P_I) \setminus O_I\}$$

$$= P'_1 \cup P'_2 \cup (-P_I) \setminus O_I.$$
Proposition (1)

The optimal set \{P_1', P_2'\} has the smallest difference in length \( Y = l(P_1') + l(P_2') - l(P_1) \geq 0 \) from the shortest path \( P_1 \), among all the possible sets of arc-disjoint path pairs.

Proof.

We will prove this proposition by using the definition of the logical difference set.

With \( l(-P_1) = -l(P_1) \), \( (P_1' \cup P_2') - P_1 = P_1' \cup P_2' \cup (-P_1) \setminus O_t \), we have
\[
l((P_1' \cup P_2') - P_1) = l((P_1' \cup P_2' \cup (-P_1) \setminus O_t) \-
\quad l((P_1' \cup P_2') \cup (-P_1)) - l(O_t) \-
\quad l(P_1' \cup P_2') + l(-P_1) \-
\quad l(P_1') + l(P_2') + l(-P_1) \-
\quad l(P_1') + l(P_2') - l(P_1) \-
\quad Y.
\]

The following lemma shows that the logical difference set forms the shortest path in the modified network.

Lemma (4)

Given a directed network \( G(V, A) \) and pair \( (s, t) \) and let \( P_1 \) be the shortest path in this network. We define \( G'(V, A') \) as the network \( G(V, A) \) for which the path \( P_1 \) is replaced with \(-P_1\). The logical difference set \( (P_1' \cup P_2') - P_1 \) between the optimal set of two arc-disjoint paths \{\( P_1', P_2' \)\} and the shortest path \( P_1 \) forms the shortest path \( P_2 \) from node \( s \) to node \( t \) in \( G'(V, A') \).

Proof.

We will first prove that \( P_2 = (P_1' \cup P_2') - P_1 \) is a complete path from \( s \) to \( t \) in \( G'(V, A') \).

From Lemma (1), the optimal set of two arc-disjoint paths \( P_1' \cup P_2' \) must contain either the first shortest path \( P_1 \) itself or some \( P_1 \) arcs on each of its two paths.
If $P'_1 \cup P'_2 \supseteq P_1$, without loss of generality, suppose $P'_1 = P_1$, then $O_1 = P_1 \cup (-P_1)$. With the definition of logical difference set, we have $P_2 = ((P'_1 \cup P'_2) \cup (-P_1)) \setminus O_1 = (P_1 \cup P'_2 \cup (-P_1)) \setminus (P_1 \cup (-P_1)) = P'_2$.

Hence $P_2$ must be a complete path from $s$ to $t$.

If $P'_1 \cup P'_2$ contains some $P_1$ arcs on each of its two paths, as $-P_1$ is the path from $t$ to $s$ in $G(V, A')$, and neither $P'_1$ nor $P'_2$ contains any $-P_1$ arcs, then the union $P'_1 \cup P'_2 \cup (-P_1)$ contains two cycles: one cycle consists of $P'_1$ and $-P_1$, the other consists of $P'_2$ and $-P_1$. When the set $O_1$ is removed from the union set, the remaining arcs compose the logical difference set $P_2$. Hence $P_2$ must be a complete path from $s$ to $t$.

Now we will prove that $P_2$ is the shortest path in $G(V, A')$. Assume that the shortest path in $G(V, A')$ is $P_3 \neq P_2$, then we must have $l(P_3) < l(P_2)$. As $l(P_2) = l(P'_1) + l(P'_2) - l(P_1)$ we have $l(P_3) < l(P'_1) + l(P'_2) - l(P_1)$ and $l(P_3) + l(P_1) < l(P'_1) + l(P'_2)$, which contradicts the assumption that $\{P'_1, P'_2\}$ is the optimal set.

Lemma (5)

$P_j$ is a shortest path from $1$ to $j$ in $G$ and $(i, j)$ is the last arc of $P_j$ if and only if $P_i$ which can be obtained by dropping $(i, j)$ from $P_j$ is a shortest path from 1 to $i$.

![Fig. 4. Paths $P_j$ and $P_i$](image-url)
Proof.

Suppose that $P_j$ is a shortest path from 1 to $j$. Assume $P_i$ is not a shortest path from 1 to $i$. Then there is a path $P_i^*$ that is shorter than $P_i$. Hence if we now add $(i, j)$ to $P_i^*$, we get a path from 1 to $j$ that is shorter than $P_j$. This contradicts our assumption that $P_j$ is shorter. Therefore $P_i$ is a shortest path from 1 to $i$.

Conversely, suppose that $P_i$ is a shortest path from 1 to $i$. Obviously, $P_j$ is a shortest path from 1 to $j$ because $P_j$ is the path which can be obtained by adding $(i, j)$ to a shortest path $P_j$.

Many routing algorithms assume non-negative arc lengths to avoid a cycle of negative length appearing on a path. However, negative arc lengths introduced to a network in Bhandari’s algorithm will not cause cycles in the routing process.

**Theorem (1)**

Given a directed network $G(V, A)$ and source-destination pair $(s, t)$ and let $P_1$ be the shortest path in this network. The modified network $G(V, A')$ is defined as the network $G(V, A)$ for which $P_1$ is replaced with $-P_1$. A cycle containing some negative arcs in $G(V, A')$ will not have a negative length.

---

**Fig. 5.** A cycle contains some negative arc;
(a) The shortest path $P_1(s, t)$; and (b) A cycle containing some $-P_1$ arc
Proof.

Assume $s v_1 ... v_i v_{i+1} ... v_n t$ is the shortest path $P_1$ from node $s$ to node $t$ in $G (V, A)$, as shown in Fig. 5 (a). The corresponding path $-P_1$ in $G (V, A')$ (Fig. 5 (b)) has an arc $(v_{i+1}, v_i)$ which appears on cycle $B_1 = u_i v_{i+1} v_i u_i$.

Suppose the cycle $B_1$ has a negative length $l(B_1) = w_{u_i v_{i+1}} + w_{v_i v_{i+1}} + w_{v_i u_i} < 0$. Because $w_{v_i v_{i+1}} = -w_{v_i v_{i+1}}$, we must have $w_{u_i v_{i+1}} - w_{v_i v_{i+1}} + w_{v_i u_i} < 0$ and $w_{v_i u_i} + w_{u_i v_{i+1}} < w_{v_i v_{i+1}}$. Hence the sub-path $s v_1 ... v_i u_i v_{i+1}$ is shorter than the sub-path $s v_1 ... v_i v_{i+1}$.

This contradicts the assumption that $s v_1 ... v_i v_{i+1} ... v_n t$ is the shortest path.

The following theorem gives the optimality of the solution produced with Bhandari’s algorithm.

**Theorem (2)**

Given a directed network $G (V, A)$ and source-destination pair $(s, t)$, the Bhandari’s algorithm returns the optimal set for the APP problem.

**Proof.**

Let $P_1$ be the shortest path in the original network $G (V, A)$ found in Step 1 of Bhandari’s algorithm and $P_2$ be the shortest path in the modified network $G (V, A')$, found in Step 3 of Bhandari’s algorithm. Let $\{P'_1, P'_2\}$ be the solution set generated by Bhandari’s algorithm. We will prove that $\{P'_1, P'_2\}$ has the following properties.

(i) By construction of the solution set, we must have $P'_1 \cap P'_2 = \emptyset$. So the required path pair $\{P'_1, P'_2\}$ is arc-disjoint.

(ii) Suppose the optimal set of arc-disjoint paths is $\{P''_1, P''_2\}$ instead of $\{P'_1, P'_2\}$. According to Lemma (4), the logical difference set of $\{P''_1, P''_2\}$ with $P_1$ is the shortest path in the modified network $G (V, A')$. This
contradicts that $P_2$ is the shortest path in modified network $G (V, A')$. So the solution set $\{P_1', P_2'\}$ generated by Bhandari’s algorithm has minimal total length.

(iii) On Theorem (1), a cycle in the modified network $G (V, A')$ will not have a negative length.

By the above properties, the solution set returned by Bhandari’s algorithm must be the optimal set.

**Conclusion**

In this paper Bhandari’s algorithm which produces an optimal solution for (APP) problem has been described. We first showed that the optimal set for the (APP) problem is based on the shortest path. Secondly, we showed that the optimal set of two arc-disjoint path has the smallest difference in length from the shortest path among all the possible set of arc-disjoint paths. And we also proved that the logical difference set forms the shortest path in the modified network. Finally we proved that the solution produced with Bhandari’s algorithm is optimal.

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